Proposed SciHarbor Task:
Upper Critical Dimension of Spin Glass Models in a Field

Gangmin Son
Korea Institute for Advanced Study

June 12, 2026

Abstract

This document outlines a proposed benchmark task for SciHarbor, focusing on a resilient open problem in statistical physics: determining the upper critical dimension of spin-glass models in an external magnetic field. We provide the physical formalism and contextual background to frame this analytical and numerical intractability as a rigorous testbed for evaluating advanced scientific reasoning.

1 Overview

Determining the exact upper critical dimension D_U of spin glass models in a magnetic field remains a formidable challenge. The difficulty is twofold: (i) exact analytical methods break down in finite dimensions, and (ii) numerical simulations are heavily restricted by finite-size effects and massive equilibration times. Although a recent framework [Phys. Rev. Lett. **128**, 075702 (2022)] suggests $D_U \geq 8$, the exact verification of the value still remains open.

To contextualize this challenge, we first introduce the basic principles of phase transitions and mean-field theory. Subsequently, we detail the specific theoretical breakdown introduced by the spin-glass phase under an external field.

2 Phase Transitions

The **Ising model** is a minimal model for magnetism. Each site i on a D-dimensional hypercubic lattice carries a spin $s_i \in \{+1, -1\}$, representing a tiny magnetic moment pointing up or down. Neighboring spins interact with coupling strength $J > 0$, and an external magnetic field h biases all spins. The energy (Hamiltonian) of the system is

$$\mathcal{H} = -J \sum_{\langle i,j \rangle} s_i s_j - h \sum_i s_i, \tag{1}$$

where the first sum runs over all nearest-neighbor pairs.

At temperature T, the system samples configurations with probability proportional to $e^{-\mathcal{H}/k_B T}$ (the Boltzmann distribution). High T means thermal fluctuations dominate and spins are disordered; low T means the energy term dominates and spins tend to align.

The degree of alignment is captured by the **order parameter**, the magnetization:

$$m = \frac{1}{N} \sum_i \langle s_i \rangle, \tag{2}$$

where $\langle \cdot \rangle$ denotes a thermal average. At zero field, $m = 0$ in the disordered (paramagnetic) phase and $m \neq 0$ in the ordered (ferromagnetic) phase. The two phases are separated by a **phase transition** at a critical temperature T_c.

Near T_c, physical quantities follow power laws in $|T - T_c|$ with exponents called **critical exponents**. These exponents depend only on broad features of the system (dimensionality, symmetry), not microscopic details—a phenomenon known as **universality**.

3 Mean-Field Theory and Upper Critical Dimension

Mean-field theory (MFT) simplifies the many-body problem by replacing all interactions on a given spin with a single effective field produced by the average behavior of its neighbors. This approximation becomes exact when every spin interacts equally with every other spin, corresponding to a fully connected graph or, equivalently, an infinite-dimensional lattice ($D \to \infty$).

In finite dimensions, fluctuations ignored by MFT become increasingly important as D decreases. The **upper critical dimension** D_U is defined as the spatial dimension below which fluctuations are strong enough to change the critical exponents away from their mean-field values. For $D > D_U$, MFT gives the correct exponents; for $D < D_U$, it does not.

For the standard ferromagnetic Ising model, $D_U = 4$: MFT is exact in four or more dimensions, while the classic exponents (e.g., $\beta = 1/8$ in 2D) differ from MFT predictions for $D < 4$.

4 Spin Glasses and the Open Problem

4.1 What is a Spin Glass?

A spin glass arises when the couplings J_{ij} are *random* rather than uniform, drawn independently from a distribution that allows both positive (ferromagnetic) and negative (antiferromagnetic) values, e.g., $J_{ij} \sim \mathcal{N}(0, J^2/N)$. The Hamiltonian becomes

$$\mathcal{H} = -\sum_{\langle i,j \rangle} J_{ij}\, s_i s_j - h \sum_i s_i. \tag{3}$$

Negative couplings cause **frustration**: some pairs of spins cannot simultaneously minimize their interaction energy, no matter how they orient. The result is a complex energy landscape with an exponentially large number of nearly degenerate low-energy configurations. Below a critical temperature T_c, the system freezes into one of these configurations—a **spin-glass phase**—rather than finding a unique ordered state like a ferromagnet.

4.2 The Role of an External Field

In a ferromagnet, a small external field h breaks the symmetry immediately: there is no true phase transition for $h \neq 0$, only a smooth crossover. Spin glasses are fundamentally different. The mean-field theory (the Sherrington–Kirkpatrick model) predicts that a genuine phase transition *persists* in a field, along the so-called **de Almeida–Thouless (AT) line** in the (T, h) plane. Above this line lies the paramagnetic phase; below it, the spin-glass phase. Whether the AT transition survives in finite-dimensional systems is one of the central open questions in the field.

4.3 The Upper Critical Dimension Problem

Even accepting the existence of the AT transition in high enough dimensions, determining D_U precisely is very hard:

- **Analytical difficulty.** The standard renormalization-group (RG) expansion around MFT breaks down for spin glasses in a field in ways not fully understood. Unlike the ferromagnetic case, multiple RG fixed points and subtle symmetry considerations lead to conflicting predictions.
- **Numerical difficulty.** Equilibrating a spin glass requires exponentially long simulation times as $T \to T_c$. Direct simulations at $D = 5, 6$ are near the boundary of what is computationally feasible, and results remain ambiguous.

A promising workaround is provided by **one-dimensional long-range spin glass models**, where spins on a line interact with a coupling that decays as $J_{ij} \sim |i - j|^{-\sigma}$. By tuning the exponent σ, one can effectively control the "dimension" of the model, making it possible to probe the AT transition across a continuous range of effective D without changing the actual geometry. This strategy allows finite-size scaling studies to be performed at a much lower computational cost than in literal high-dimensional lattices. A recent study [Angelini et al., Phys. Rev. Lett. **128**, 075702 (2022)] used this approach and found evidence that $D_U \geq 8$.